CRITICAL PATH ANALYSIS

Critical Path Analysis breaks down a complex task into component activities to optimise performance

Any task involving more than a single operation can be analysed into sets of component activities. Examples are: building operations, development of space exploration programme, writing a book etc. The common factor is that each consists of many activities or subprojects. Success or failure of the project results from the management and scheduling of these activities. Information on likely activity durations, costs and logically necessary sequences can be used to optimise project performance. Methods of network analysis (Critical Path Analysis, Programme Evaluation and Review Technique etc) were developed in the 1950s in the UK by the ESI for power station construction and in the US for the Polaris programme. They are designed to identify and eliminate bottlenecks by determining critical activities, clarifying responsibility and improving control.

Some activities depend on others: they can only begin when other activities have been completed .

Other activities can take place concurrently.

The whole project is only complete when all activities have been completed.

Each activity has a duration:

- in CPA this is assumed known with certainty

- in PERT the duration has a probability distribution

Some activities in the project will be *critical*:
if these are delayed at all then the whole project is delayed

Other activities are non-critical:
these can be delayed a certain amount without delaying the whole project

Example: The Optimisers Breakfast

Step 1: Make a list of activities, durations and precedence (if any)

Activity	Precedence	Duration
A Get up	-	30
B Make tea	D, E	3
C Fill kettle	A	1
D Fetch milk	A	1
E Boil water	C	4
F Milk on cereal	D	1
G Make toast	H	4
H Cut bread	A	2
I Eat	B, F, G	10

Step 2: Develop a simple network which sets out the relationships amongst these activities and distinguishes SEQUENTIAL (e.g. A to C) from CONCURRENT activities (e.g. C D and H). This is the fundamental analysis which enables a rational program to be developed.

A network diagram of the optimisers breakfast is shown below:
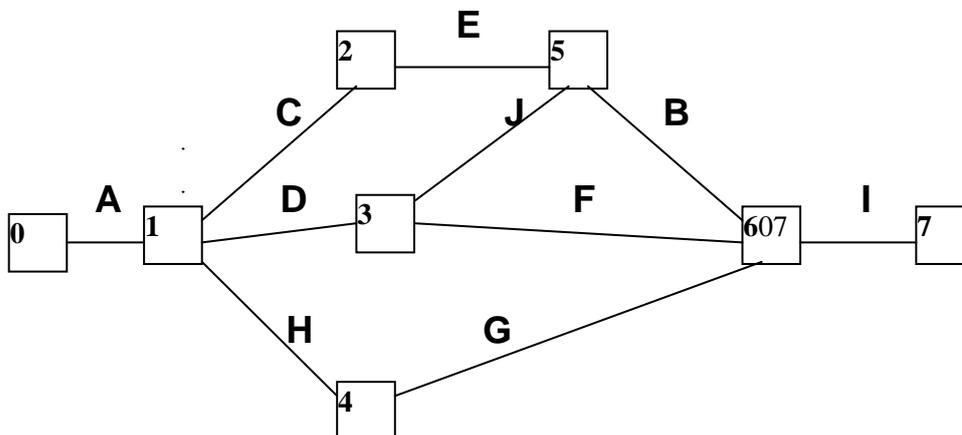
In the network diagram, activities are represented by arrows (labelled by letters) while *events* are represented by nodes (circles) and labelled by numbers.

Events are points in time marking the completion of activities and are represented as nodes in the network. Events are numbered serially. Thus event 0 is the node before A and event 1 is the node

after A. No event can be numbered until all preceding events have been numbered.

Two nodes can be linked by at most one activity. Thus the following is an incorrect representation:
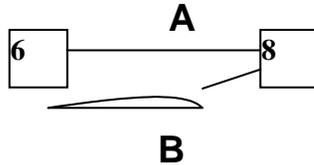
A correct representation would be given by
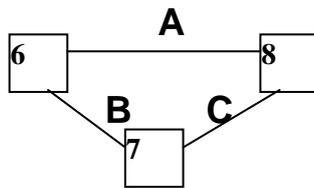
which introduces a new node (7) and a *dummy activity* C linking nodes 7 and 8.

Dummy Activities are inserted where precedence needs to be indicated but no time or other resources are consumed. For example, before tea can be made (B), water must be boiled (E) and milk fetched (D). To indicate this on the diagram a dummy activity needs to be introduced between nodes 3 and 5. This activity (J) has zero duration.

Duration. In order to optimise the project, information on time duration of activities is required. This time is likely to be inversely related to the level of resources input. Initially we will make an estimate based on a current resource input. Later we will allow for potential uncertainty in duration estimates.

The Critical Path is the longest path of necessarily sequential activities within the project. In the example there are four paths: A-C-E-B-I, A-D-J-B-I, A-D-F-I and A-H-G-I. The problem is how to determine the lengths of the paths.

The following concepts are useful:

Earliest Starting and Finishing Times The earliest starting time for activity i, ES_i. In the example $ES_a = 0$, $ES_c = ES_d = ES_h = 30$ etc. The earliest finishing time for activity i, EF_i, is linked to ES_i by

$EF_i = ES_i + T_i$ where T_i is estimated time of completion of activity i. ES_i and EF_i are determined by a forward pass through the network. Where more than one activity ends in a node, the ES_i for activities emanating from this node is the largest EF_i of the activities ending in the node.

Latest Starting and Finishing Times The latest starting time for activity i, LS_i is the latest time at which activity i can be started without delaying the project. The latest finishing time for activity i, LF_i, is the latest time that an activity can be completed without delaying the project. $LS_i = LF_i - T_i$. LS_i and LF_i are determined by a backward pass through the network. Where more than one activity begins in the same node, the smaller of the LS_is will determine the LF_i of activities terminating in the node.

Activity Slack, S_i, is the time available for 'fine tuning' the project. On any activity i it is equal to $LF_i - EF_i$ which is the same as $LS_i - ES_i$.

Critical Path The critical path is the path (or paths) along which the slack on all activities is zero, i.e. it is the longest necessary path through the network.

To see the relationships it is helpful to set up the problem in a tableau with columns for activities, preceding activities and durations. Then the various statistics can easily be evaluated:

Act	Prec	Ti	ESi	LSi	EFi	LFi	Si	Critical Path
A	-	30	0	0	30	30	0	Yes
B	D, E	3	35	35	38	38	0	Yes
C	A	1	30	30	31	31	0	Yes
D	A	1	30	34	31	35	4	No
E	C	4	31	31	35	35	0	Yes
F	D	1	31	37	32	38	6	No
G	H	4	32	34	36	38	2	No
H	A	2	30	32	32	34	2	No
I	B, F,G	10	38	38	48	48	0	Yes

CRITICAL PATH: A - C - E - B - I
NETWORK COMPLETION TIME = 48

PROGRAM EVALUATION & REVIEW

Handling Uncertainty

In the simple model developed above we have assumed certainty in activity lengths. However, once uncertainty is introduced, the critical path length will depend in some way on combined uncertainty in the activities. If we know the individual activity variances and can assume they are independent of each other, then the variance of the project is simply the sum of the variances along the critical path. However, if the variances are large it is conceivable that some other path could become critical and the whole uncertainty analysis would become intractable. The most common assumption is that uncertainty follows the beta distribution. This makes the uncertainty of duration of each activity a function of three estimates (guesses) : the most likely duration, an optimistic estimate of duration and a pessimistic estimate of duration. The beta distribution is unimodal and has fixed end points. The critical path variance is calculated and for big enough problems the normal distribution can be used to find the probability of completing the project in a specified time.

Assumptions of PERT

(1)   The uncertainties in duration of events are all *independent*

This assumption implies

(a)   Expected duration of path is sum of expected duration of all activities along the path
(b)   Variance of path is sum of variances of all activities along the path

(2)   The distribution of the path duration is *normal*
This will follow from invoking a central limit theorem but this requires the number of activities to be large enough

(3)   Critical path *remains critical* despite uncertainty

Without this assumption the variance is very difficult to calculate.

The Beta Distribution

The Beta distribution is a probability distribution f(x).

It is *unimodal* and *assymetric*

It can be expressed as a function of 3 parameters: m, a and b

m   most likely completion time
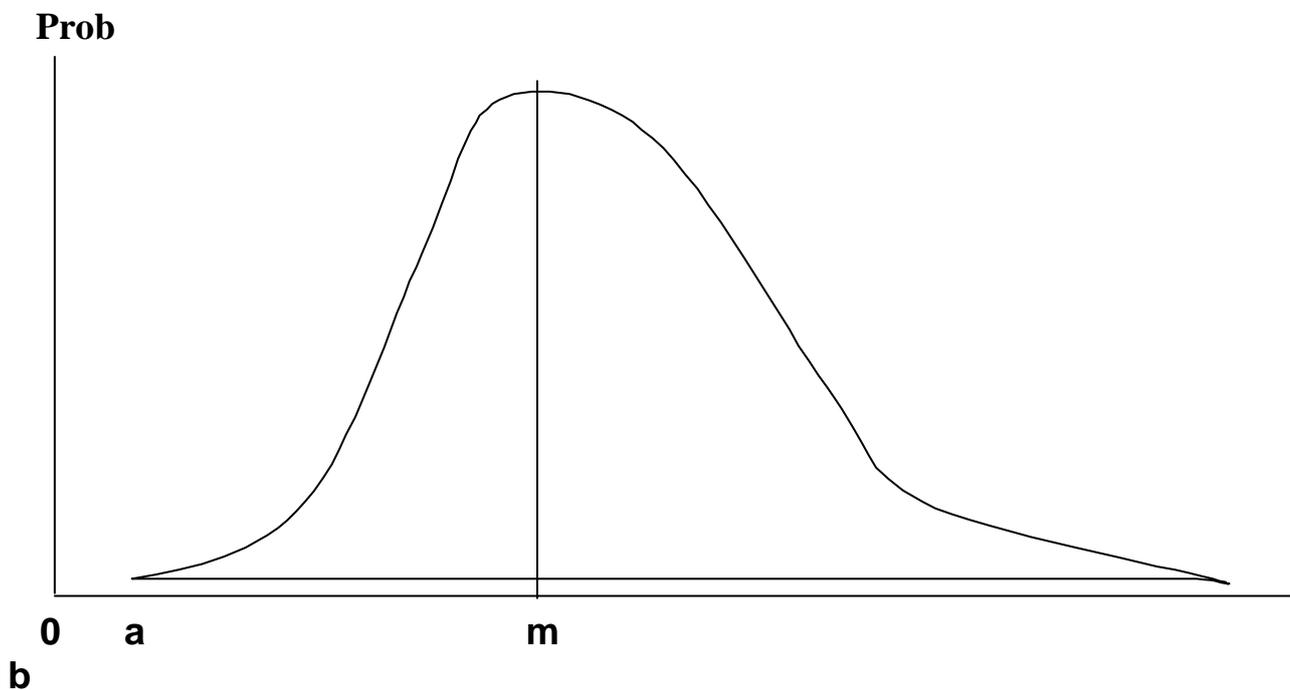a   optimistic completion time
b   pessimistic completion time

Expected completion time is given by:

$$E(x) = (a + 4m + b) / 6$$

Variance of completion time is given by:

$$Var(x) = [(b - a) / 6]^2$$

The Beta Distribution

Costs The duration of each activity in a project is likely to be a function of the resources that are put into that activity. By putting in more resources, the duration can be reduced but at increased cost. One common question is how can a project be shortened to a specified length at minimum cost. This involves calculating the marginal cost of shortening each activity, and then reducing the lowest MC activity on the critical path until its MC exceeds the next lowest MC activity and so on until the target reduction is achieved, bearing in mind any constraints on the extent of reductions and the lengths of non critical paths which may become critical if the existing path is reduced far enough. This is not a trivial exercise.